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Holonomic deformation of linear differential equations of the A_3 type and polynomial Hamiltonian structure

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Abstract

We study the theory of the holonomic deformation for the linear differential equation of the A_3 type and show that the holonomic deformation of the equation is represented by the Hamiltonian system with the polynomial Hamiltonian. We also give the particular solutions of the polynomial Hamiltonian system.

0 Introduction.

In this paper, we consider following linear differential equation with irregular singular point, $x = \infty$ and three non logarithmic singular points, $x = \lambda_1, \lambda_2, \lambda_3$,

$$\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)y = 0, \tag{0.1}$$

defined on the Riemann scheme $\mathbf{P}^1$, such that

$$\begin{aligned} p_1(x) &= -2x^4 - \sum_{(j)} j s_j x^{j-1} - \frac{3}{4}s_3^2 - \sum_{(k)} \frac{1}{x-\lambda_k}, \\ p_2(x) &= -(2\alpha+1)x^3 - 2\sum_{(j)} H_j x^{3-j} + \sum_{(k)} \frac{\mu_k}{x-\lambda_k}. \end{aligned} \tag{0.2}$$

We suppose that $2\alpha + 1$ is not an integer through out this paper, This equation has an irregularity at $x = \infty$ of the Poincaré rank 5, and three regular regular points $x = \lambda_k$ $(k = 1, 2, 3)$. We make the following assumption:

(A) None of $x=\lambda_k$ $(k=1,2,3)$ is logarithmic singularity.

The aim of this paper is to study the holonomic deformation of (0.1)–(0.2) under the assumption (A). The main results of this paper are as follows :

Main theorem. *Under the assumption (A), the holonomic deformation of the equation* (0.4) *is governed by the completely integrable Hamiltonian system of partial differential equations :*

$$(H)\qquad \begin{aligned}\frac{\partial\lambda_k}{\partial s_j}&=\frac{\partial H_j}{\partial\mu_k}\\ \frac{\partial\mu_k}{\partial s_j}&=-\frac{\partial H_j}{\partial\lambda_k}\end{aligned}\qquad (k,j=1,2,3).$$

The linear equation (0.1)–(0.2) is a particular case of an equation of the form (0.1), such that

$$(0.3)\qquad \begin{aligned}p_1(x)&=-2x^{g+1}-\sum_{j=1}^{g}jt_jx^{j-1}-\sum_{k=1}^{g}\frac{1}{x-\lambda_k},\\ p_2(x)&=-(2\alpha+1)x^g-2\sum_{j=1}^{g}H_jx^{g-j}+\sum_{k=1}^{g}\frac{\mu_k}{x-\lambda_k}.\end{aligned}$$

with the following Riemann scheme:

$$(0.4)\qquad \begin{pmatrix} x=\lambda_k & \overbrace{\hphantom{aaaaaaaaaaaaaaaaaaaaaaaa}}^{x=\infty} \\ \begin{matrix}0\\2\end{matrix} & \begin{matrix}0 & 0 & 0 & 0 & \cdots & 0 & \alpha+\frac{1}{2}\\ \frac{2}{g+2} & 0 & t_g & t_{g-1} & \cdots & t_1 & -\alpha+\frac{1}{2}\end{matrix}\end{pmatrix}\qquad (k=1,\cdots,g)$$

Here the symbol in (0.4) means that, at the irregular point $x=\infty$, the equation (0.1)–(0.2) admits a system of formal solutions of the form:

$$(0.5)\qquad \begin{aligned}y_1&=x^{-\frac{1+2\alpha}{2}}\sum_{n\geq 0}h_{1,n}x^{-n},\\ y_2&=x^{\frac{2\alpha-1}{2}}\exp\{\frac{2}{5}x^5+\sum_{(i)}s_ix^i+\frac{3}{4}{s_3}^2x\}\sum_{n\geq 0}h_{2,n}x^{-n}.\end{aligned}$$

Note that the Poincaré rank at $x=\infty$ of the linear equation (0.1)–(0.3) is $g+2$. The principal parts of the formal solutions are given by the primitive function of the polynomial representing the versal deformation of the simple singularity of the A_g type, so we call the linear equation (0.1)–(0.3) as the equation of A_g type.

When considering the holonomic deformation of equation of the A_1 type, we obtain the Hamiltonian structure

$$(\lambda_1, \mu_1, H_1, t_1).$$

which determines the Hamiltonian system, equivalent to the second Painlevé equation, see [1].

When $g = 2$, it is known [2] that the holonomic deformation is governed by the Hamiltonian system with respect to the canonical variables:

$$(\lambda_1, \lambda_2, \mu_1, \mu_2, H_1, H_2, t_1, t_2).$$

On the other hand, in the case $g \geq 3$, the quantities $H = (H_1, \cdots, H_g)$ and $t = (t_1, \cdots, t_g)$ do not compose the Hamiltonian structure. In fact, even in the case $g = 3$, we have to determine the variables $s = (s_2, s_2, s_3)$ such that

$$s_1 = t_1 + \frac{3}{4}t_3^2, \qquad s_2 = t_2, \qquad s_3 = t_3, \tag{0.6}$$

and then obtain the Hamiltonian structure

$$(\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3, H_1, H_2, H_3, s_1, s_2, s_3).$$

We are studying in the present paper the holonomic deformation of equation of the A_3 type. As to the case of $g \geq 4$, we will study in the future.

1 Holonomic deformation of linear equation of the second order.

In this section, we recall the theory of the holonomic deformation of linear differential equation of the form:

$$\frac{d^2y}{dx^2} + p_1(x,s)\frac{dy}{dx} + p_2(x,s)y = 0, \tag{1.1}$$

viewing $s = (s_1, \cdots, s_g)$ as the deformation variables.

Proposition 1.1. *the equation* (1.1) *has a fundamental system of solutions whose monodromy and Stokes multipliers are independent of the parameter s, if and only if there exist rational functions of x,* $A_j(x), B_j(x)$*, such that the following system of partial differential equations is completely integrable.*

$$\begin{aligned} &\frac{\partial^2 y}{\partial x^2} + p_1(x,s)\frac{\partial y}{\partial x} + p_2(x,s)y = 0, \\ &\frac{\partial y}{\partial s_j} = B_j(x)y + A_j(x)\frac{\partial y}{\partial x}. \end{aligned} \tag{1.2}$$

Proposition 1.2. *The conditions of the complete integrability of* (1.2) *are given by:*

$$\frac{\partial A_j}{\partial s_i} + A_j\frac{\partial A_i}{\partial x} = \frac{\partial A_i}{\partial s_j} + A_i\frac{\partial A_j}{\partial x} \qquad (i,j = 1, cdots, g). \tag{1.3}$$

$$A_j\frac{\partial^3 A_j}{\partial x^3} - 2\frac{\partial}{\partial x}(A_j^2 P) + 2A_j\frac{\partial P}{\partial t_j} = 0. \qquad (j = 1, \cdots, g). \tag{1.4}$$

where

$$P(x,s) = -p_2(x,s) + \frac{1}{4}p_1^2(x,s) + \frac{1}{2}\frac{\partial}{\partial x}p_1(x,s). \tag{1.5}$$

If we make the change of the unknown function:

$$\begin{aligned} &y = \Phi(x)z, \\ &\Phi(x) = \exp(-\frac{1}{2}\int^x p_1(x,t)\,dx). \end{aligned} \tag{1.6}$$

then (1.1) is transformed to an equation of the form:

$$\frac{d^2 z}{dx^2} = P(x,s)z. \tag{1.7}$$

Proposition 1.3. *The holonomic deformation of* (1.1) *is reduced to that of* (1.7).

A linear equation of the form (1.7) is called the SL-type equation.

Finally the holonomic deformation of (1.1) is reduced to the existence of rational functions $A_j(x)$, satisfying the system (1.3)–(1.4) of partial differential equations. We will call (1.2) as the extended system of (1.1) and the functions $A_j(x)$ as the deformation functions.

2 Deformation function $A_j(x)$.

In the following of this paper, we consider the holonomic deformation of linear equations of the form:

$$\frac{d^2y}{dx^2} + p_1(x,s)\frac{dy}{dx} + p_2(x,s)y = 0, \tag{2.1}$$

$$\begin{aligned} p_1(x) &= -2x^4 - \sum_{(j)} j s_j x^{j-1} - \frac{3}{4}s_3^2 - \sum_{(k)} \frac{1}{x-\lambda_k}, \\ p_2(x) &= -(2\alpha+1)x^3 - 2\sum_{(j)} H_j x^{3-j} + \sum_{(k)} \frac{\mu_k}{x-\lambda_k}, \end{aligned} \tag{2.2}$$

under the assumption (A). For the limiting pages, here we only can give the results, omit their proofs.

Firstly we determine the deformation functions.

Proposition 2.1. *The function $A_j(x)$ ($j = 1,2,3$) are given as follows:*

$$A_j(x) = \frac{Q_{j-1}(x)}{\Lambda(x)}, \tag{2.3}$$

where

$$Q_{j-1}(x) = \sum_{i=0}^{j-1} q_i^{(j)}(s)x^{j-1-i}, \quad \Lambda(x) = \prod_{j=1}^{3}(x-\lambda_j).$$

The explicit form of $q_i^{(j)}(s)$ ($j = 1,2,3; i = 0,\cdots,j-1$) will be given in Section 4

In order to prove this proposition, we shall first study the singularity of $A_j(x)$.

Lemma 2.1. *For any fixed s, each $A_j(x)$ is holomorphic on* $\mathbf{C}\setminus\{\lambda_1,\lambda_2,\lambda_3\}$.

Lemma 2.2. *For $k = 1,2,3$, $x = \lambda_k$ is a pole of the first order of $A_j(x)$.*

Lemma 2.3. *$A_j(x)$ admits a zero of order $4-j$ at $x = \infty$.*

3 Equations of the SL-type.

In this section, we will investigate the equation of the SL-type:

$$
\begin{aligned}
&\frac{d^2z}{dx^2} = P(x,s)z,\\
&P(x,s) = -p_2(x,s) + \frac{1}{4}p_1^2(x,s) + \frac{1}{2}\frac{\partial}{\partial x}p_1(x,s).
\end{aligned}
\tag{3.1}
$$

$P(x,s)$ can be written in the following form:

$$
\begin{aligned}
P(x,s) = x^8 + x^3\sum_{j=0}^{3} F_j x^j + 2\sum_{j=1}^{3} K_j x^{3-j}\\
-\sum_{k=1}^{3}\frac{\nu_k}{x-\lambda_k} + \frac{3}{4}\sum_{k=1}^{3}\frac{1}{(x-\lambda_k)^2},
\end{aligned}
\tag{3.2}
$$

where, we have following relations:

$$
F_3 = 3s_3, \qquad F_2 = 2s_2, \qquad F_1 = s_1 + 3s_3^2, \qquad F_0 = 2\alpha + 3s_2s_3.
$$

$$
\begin{aligned}
K_1 &= H_1 + \frac{3}{4}s_3(s_1 + \frac{3}{4}s_3^2) + \frac{1}{2}s_2^2 + \frac{1}{2}\sum_{k=1}^{3}\lambda_k,\\
K_2 &= H_2 + \frac{3}{4}s_3 + \frac{1}{2}s_2(s_1 + \frac{3}{4}s_3^2) + \frac{1}{2}\sum_{k=1}^{3}\lambda_k^2,\\
K_3 &= H_3 + s_2 + \frac{1}{8}(s_1 + \frac{3}{4}s_3^2)^2 + \frac{1}{2}\sum_{k=1}^{3}\lambda_k^3 + \frac{3}{4}s_3\sum_{k=1}^{3}\lambda_k.
\end{aligned}
\tag{3.3}
$$

$$
\nu_k = \mu_k - \frac{1}{2}\sum_{l=1;\neq k}^{3}\frac{1}{\lambda_k-\lambda_l} - \frac{1}{2}\sum_{i=1}^{3} i s_i \lambda_k^{i-1} - \frac{3}{8}s_3^2 - \lambda_k^4.
\tag{3.4}
$$

For the simplicity of presentation, we put:

(3.5)

$$
N_k = \frac{1}{\prod_{i=1;\neq k}^{3}(\lambda_k-\lambda_i)}, \quad N^{jk} = (-1)^{j-1}\sigma_{j-1}(\check{\lambda}_k), \quad \Lambda(x) = \prod_{i=1}^{3}(x-\lambda_i),
$$

where $\sigma_j(\check{\lambda}_k)$ denotes the j-th elementary symmetric polynomials of two variables, λ_j $(j = 1, 2, 3; \neq k)$.

Proposition 3.1. *The Hamiltonian functions* H_j $(j = 1, 2, 3)$ *are given by*

$$H_j = \frac{1}{2}\sum_{(k)}[N_k N^{jk}\mu_k^2 - U_{jk}\mu_k - N_k N^{jk}(2\alpha+1)\lambda_k^3], \tag{3.6}$$

where

$$U_{jk} = N_k N^{jk}(2\lambda_k^4 + 3s_3\lambda_k^2 + 2s_2\lambda_k + s_1 + \frac{3}{4}s_3^2) - \sum_{l=1;\neq k}^{3} \frac{N_k N^{jk} + N_l N^{jl}}{\lambda_l - \lambda_k}.$$

Proposition 3.2. *In the linear equation* (3.1)-(3.2), K_j $(j = 1, 2, 3)$ *are written as follows:*

$$K_j = \frac{1}{2}\sum_{k=1}^{3}(N_k N^{jk}\nu_k^2 - \sum_{l=1;\neq k}^{3} \frac{N_l N^{jl}}{\lambda_k - \lambda_l}\nu_k - N_k N^{jk} V_k), \tag{3.7}$$

where

$$V_k = \lambda_k^8 + \lambda_k^3 \sum_{j=0}^{3} F_j \lambda_k^j + \frac{3}{4}\sum_{l=1;\neq k)}^{3} \frac{1}{(\lambda_k - \lambda_l)^2}.$$

Now we define $\overline{K}_j$ (j=1, 2, 3) by

$$\overline{K}_j = K_j - \frac{1}{2}(s_3^3 + s_1 s_3 + \frac{1}{2}s_2^2)\delta_{j1} - \frac{3}{8}s_2 s_3^2 \delta_{j2} - \frac{1}{4}s_2\delta_{j3} \qquad (j = 1, 2, 3),$$

where δ_{ij} is the Kronecker's delta.

Proposition 3.3. *The transformation*

$$(s, \lambda, \mu, H) \to (s, \lambda, \nu, \overline{K}) \qquad (j, k = 1, 2, 3)$$

is canonical. where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, $\mu = (\mu_1, \mu_2, \mu_3)$, $H = (H_1, H_2, H_3)$, $\nu = (\nu_1, \nu_2, \nu_3)$, $\overline{K} = (\overline{K}_1, \overline{K}_2, \overline{K}_3)$, *and* $s = (s_1, s_2, s_3)$.

4 The $A_3(x)$ system.

In this section, we will prove Main Theorem. By means of Propositions 1.2, 1.3 and 3.3, it suffices to establish the following theorem:
Theorem 4.1. *The conditions* (1.3), (1.4) *of the complete integrability are equivalent to the following completely integrable Hamiltonian system of partial differential equations:*

$$(\overline{K}) \qquad \frac{\partial \lambda_k}{\partial s_j} = \frac{\partial \overline{K}_j}{\partial \nu_k} \qquad \frac{\partial \nu_k}{\partial s_j} = -\frac{\partial \overline{K}_j}{\partial \lambda_k} \qquad (k, j = 1, 2, 3).$$

Lemma 4.1.

$$\begin{aligned} A_1(x) &= \frac{1}{2}\frac{1}{\Lambda(x)}, \qquad A_2(x) = \frac{1}{2}(x - \sigma_1)\frac{1}{\Lambda(x)}, \\ A_3(x) &= \frac{1}{2}(x^2 - \sigma_1 x + \sigma_2)\frac{1}{\Lambda(x)}. \end{aligned}$$

where σ_j denotes the j-th elementary symmetric polynomials of three variables, $\lambda_1, \lambda_2, \lambda_3$.

Lemma 4.2. *$(\overline{K})$ is complete integrable.*

5 The polynomial Hamiltonian structure.

Theorem 5.1. *For the Hamiltonian system (H), define the changes of the variables*

$$(5.1) \qquad \overline{\sigma}_j = (-1)^{j-1}\sigma_j, \qquad H_j = R_j \quad (j = 1, 2, 3).$$

$$(5.2) \qquad \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = \begin{pmatrix} 1 & -(\lambda_2 + \lambda_3) & \lambda_2\lambda_3 \\ 1 & -(\lambda_1 + \lambda_3) & \lambda_1\lambda_3 \\ 1 & -(\lambda_1 + \lambda_2) & \lambda_1\lambda_2 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix}.$$

The changes (5.1)*and* (5.2) *take* (H) *to the system:*

$$(R) \qquad \frac{\partial \overline{\sigma}_k}{\partial s_j} = \frac{\partial R_j}{\partial \rho_k} \quad \frac{\partial \rho_k}{\partial s_j} = -\frac{\partial R_j}{\partial \overline{\sigma}_k}$$

with the polynomial Hamiltonian of the form:

$$\vec{R} = \vec{\Sigma}^{(2)}\vec{\rho}^{(2)} + \vec{\Sigma}^{(1)}\vec{\rho}^{(1)} + \vec{\Sigma}^{(0)}\vec{\rho}^{(0)} + (\alpha + \frac{1}{2})\vec{\Sigma}. \tag{5.3}$$

where

$$\vec{\rho}^{(2)} = \begin{pmatrix} \rho_1^2 \\ \rho_2^2 \\ \rho_3^2 \end{pmatrix}, \quad \vec{\rho}^{(1)} = \begin{pmatrix} \rho_1\rho_2 \\ \rho_2\rho_3 \\ \rho_1\rho_3 \end{pmatrix}, \quad \vec{\rho}^{(0)} = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix}, \quad \vec{R} = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix},$$

$$\vec{\Sigma}^{(2)} = \frac{1}{2}\begin{pmatrix} 0 & 1 & -\overline{\sigma}_2 \\ 0 & -2\overline{\sigma}_1 & \overline{\sigma}_3 + \overline{\sigma}_1\overline{\sigma}_2 \\ 1 & \overline{\sigma}_1{}^2 & \overline{\sigma}_2{}^2 - \overline{\sigma}_1\overline{\sigma}_3 \end{pmatrix},$$

$$\vec{\Sigma}^{(1)} = \begin{pmatrix} 0 & -\overline{\sigma}_1 & 1 \\ 1 & \overline{\sigma}_1{}^2 & -\overline{\sigma}_1 \\ -\overline{\sigma}_1 & \overline{\sigma}_1\overline{\sigma}_2 + \overline{\sigma}_3 & -\overline{\sigma}_2 \end{pmatrix},$$

$$\vec{\Sigma}^{(0)} = -\frac{1}{2}\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix},$$

where

$$\begin{aligned}
\alpha_{11} &= 2\overline{\sigma}_1{}^2 + 2\overline{\sigma}_2 + 3s_3, \\
\alpha_{12} &= 2\overline{\sigma}_1\overline{\sigma}_2 + 2\overline{\sigma}_3 + 2s_2, \\
\alpha_{13} &= 2\overline{\sigma}_1\overline{\sigma}_3 + s_1 + \frac{3}{4}s_3^2, \\
\alpha_{21} &= 2\overline{\sigma}_3 + 2\overline{\sigma}_1\overline{\sigma}_2 + 2s_2, \\
\alpha_{22} &= 2\overline{\sigma}_2{}^2 + 3s_3\overline{\sigma}_2 - 2s_2\overline{\sigma}_1 + s_1 + \frac{3}{4}s_3^2, \\
\alpha_{23} &= 2\overline{\sigma}_2\overline{\sigma}_3 + 3s_3\overline{\sigma}_3 - (s_1 + \frac{3}{4}s_3^2)\overline{\sigma}_1 + 1, \\
\alpha_{31} &= 2\overline{\sigma}_1\overline{\sigma}_3 + s_1 + \frac{3}{4}s_3^2, \\
\alpha_{32} &= 2\overline{\sigma}_2\overline{\sigma}_3 + 3s_3\overline{\sigma}_3 - (s_1 + \frac{3}{4}s_3^2)\overline{\sigma}_1 + 2, \\
\alpha_{33} &= 2\overline{\sigma}_3{}^2 + 2s_2\overline{\sigma}_3 - (s_1 + \frac{3}{4}s_3^2)\overline{\sigma}_2 - \overline{\sigma}_1,
\end{aligned}$$

$$\vec{\Sigma} = -\begin{pmatrix} \overline{\sigma}_1 \\ \overline{\sigma}_2 \\ \overline{\sigma}_3 \end{pmatrix}.$$

Next we define E_1, E_2, E_3 by

$$\begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} + (\alpha + \frac{1}{2}) \begin{pmatrix} 0 \\ s_3 \\ \frac{1}{2}s_2 \end{pmatrix}.$$

The Hamiltonian system (R) is equivalent to the system:

$$(E) \qquad \frac{\partial \overline{\sigma}_k}{\partial s_j} = \frac{\partial E_j}{\partial \rho_k}, \quad \frac{\partial \rho_k}{\partial s_j} = -\frac{\partial E_j}{\partial \overline{\sigma}_k} \quad (j, k = 1, 2, 3).$$

Furthermore we define the change of the variables as follows:

$$(6.4) \qquad \begin{aligned} \overline{\sigma}_1 &= q_1, \\ \overline{\sigma}_2 &= q_2 - t_3, \\ \overline{\sigma}_3 &= q_3 + \frac{1}{2}t_3 q_1 - \frac{1}{2}t_2, \end{aligned}$$

$$(6.5) \qquad \begin{aligned} \rho_1 &= p_1 - \frac{1}{2}t_3 p_3, \\ \rho_2 &= p_2, \\ \rho_3 &= p_3, \end{aligned}$$

$$(6.6) \qquad \begin{aligned} s_1 &= t_1 - \frac{1}{4}t_3^2, \\ s_2 &= t_2, \\ s_3 &= t_3, \end{aligned}$$

$$(6.7) \qquad \begin{aligned} \overline{L}_1 &= L_1, \\ \overline{L}_2 &= L_2 - \frac{1}{2}p_3, \\ \overline{L}_3 &= L_3 + \frac{1}{2}t_3 L_1 - p_2 + \frac{1}{2}q_1 p_3. \end{aligned}$$

By simple computation, we see that the change of the variables defined by (6.4), (6.5), (6.6), (6.7) is a canonical transformation of the Hamiltonian system (E). which takes the Hamiltonian system (E) to the Hamiltonian system (L):

$$(L) \qquad \frac{\partial q_k}{\partial t_j} = \frac{\partial L_j}{\partial p_k}, \qquad \frac{\partial p_k}{\partial t_j} = -\frac{\partial L_j}{\partial q_k} \qquad (j,k = 1,2,3).$$

with the polynomial Hamiltonian:

$$(6.9) \qquad \vec{L} = \vec{Y}^{(2)}\vec{P}^{(2)} + \vec{Y}^{(1)}\vec{P}^{(1)} + \vec{Y}^{(0)}\vec{P}^{(0)} + (\alpha + \frac{1}{2})\vec{Y}.$$

where

$$\vec{L} = \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix}, \quad \vec{P}^{(2)} = \begin{pmatrix} p_1^2 \\ p_2^2 \\ p_3^2 \end{pmatrix}, \quad \vec{P}^{(1)} = \begin{pmatrix} p_1p_2 \\ p_2p_3 \\ p_1p_3 \end{pmatrix}, \quad \vec{P}^{(0)} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix},$$

$$\vec{Y}^{(2)} = \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2}q_2 \\ 0 & -q_1 & \frac{1}{2}q_1q_2 + \frac{1}{4}t_3q_1 + \frac{1}{2}q_3 - \frac{1}{4}t_2 \\ \frac{1}{2} & \frac{1}{2}q_1^2 - \frac{1}{4}t_3 & \frac{1}{2}q_2^2 - \frac{1}{2}q_1q_3 - \frac{1}{4}t_3q_1^2 + \frac{1}{4}t_2q_1 - \frac{1}{4}t_3q_2 + \frac{1}{8}t_3^2 \end{pmatrix},$$

$$\vec{Y}^{(1)} = \begin{pmatrix} 0 & -q_1 & 1 \\ 1 & q_1^2 - \frac{1}{2}t_3 & -q_1 \\ -q_1 & q_1q_2 + \frac{1}{2}t_3q_1 + q_3 - \frac{1}{2}t_2 & -q_2 \end{pmatrix},$$

$$\vec{Y}^{(0)} = -\Big(\big(Y_{ij}\big)\Big) \qquad (i,j = 1,2,3),$$

$$\begin{aligned}
Y_{11} &= q_1^2 + q_2 + \frac{1}{2}t_3, \\
Y_{12} &= q_1q_2 - \frac{1}{2}t_3q_1 + q_3 + \frac{1}{2}t_2, \\
Y_{13} &= q_1q_3 - \frac{1}{2}t_2q_1 - \frac{1}{2}t_3q_2 + \frac{1}{2}t_1, \\
Y_{21} &= q_1q_2 - \frac{1}{2}t_3q_1 + q_3 + \frac{1}{2}t_2,
\end{aligned}$$

$$\begin{aligned}
Y_{22} &= q_2^2 - t_2 q_1 - \frac{1}{2}t_3 q_2 + \frac{1}{2}t_1 - \frac{1}{4}t_3^2,\\
Y_{23} &= q_2 q_3 + (\frac{1}{4}t_3^2 - \frac{1}{2}t_1)q_1 - \frac{1}{2}t_2 q_2 - \frac{1}{2}t_2 t_3,\\
Y_{31} &= q_1 q_3 - \frac{1}{2}t_2 q_1 - \frac{1}{2}t_3 q_2 + \frac{1}{2}t_1,\\
Y_{32} &= q_2 q_3 + (\frac{1}{4}t_3^2 - \frac{1}{2}t_1)q_1 - \frac{1}{2}t_2 q_2 - \frac{1}{2}t_2 t_3,\\
Y_{33} &= q_3^2 + \frac{1}{2}t_2 t_3 q_1 - \frac{1}{2}t_1 q_2 - \frac{1}{4}t_2^2 + \frac{1}{8}t_3^3.
\end{aligned}$$

$$\vec{Y} = -\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}.$$

6 Particular solution of system (L).

Theorem 6.1. *Suppose that $\alpha = -\frac{1}{2}$ in (L).*

1) *The Hamiltonian system (L) admits a solution of the form:*

$$(q_1, q_2, q_3, p_1, p_2, p_3) = (q_1(t), q_2(t), q_3(t), 0, 0, 0). \tag{6.1}$$

$$q_j(t) = \frac{\partial}{\partial t_j}\log u \qquad (j = 1, 2, 3). \tag{6.2}$$

2) *u(t) is a general solution of the following system of partial differential*

equations:

$$
(6.3)\qquad
\begin{aligned}
\frac{\partial^2 u}{\partial t_1^2} &= -\frac{\partial u}{\partial t_2} - \frac{1}{2}t_3 u,\\
\frac{\partial^2 u}{\partial t_1 \partial t_2} &= \frac{1}{2}t_3\frac{\partial u}{\partial t_1} - \frac{\partial u}{\partial t_3} - \frac{1}{2}t_2 u,\\
\frac{\partial^2 u}{\partial t_1 \partial t_3} &= \frac{1}{2}t_2\frac{\partial u}{\partial t_1} + \frac{1}{2}t_3\frac{\partial u}{\partial t_2} - \frac{1}{2}t_1 u,\\
\frac{\partial^2 u}{\partial t_2^2} &= t_2\frac{\partial u}{\partial t_1} + \frac{1}{2}t_3\frac{\partial u}{\partial t_2} - (\frac{1}{2}t_1 - \frac{1}{4}t_3^2)u,\\
\frac{\partial^2 u}{\partial t_2 \partial t_3} &= (\frac{1}{2}t_1 - \frac{1}{4}t_3^2)\frac{\partial u}{\partial t_1} + \frac{1}{2}t_2\frac{\partial u}{\partial t_2} + \frac{1}{2}t_2 t_3 u,\\
\frac{\partial^2 u}{\partial t_3^2} &= -\frac{1}{2}t_2 t_3\frac{\partial u}{\partial t_1} + \frac{1}{2}t_1\frac{\partial u}{\partial t_2} + \frac{1}{4}(t_2^2 - \frac{1}{2}t_3^3)u.
\end{aligned}
$$

Lemma 6.2. *If $\alpha = -\frac{1}{2}$, the Hamiltonian system (L) admits a particular solution of the form:*

$$(q_1, q_2, q_3, p_1, p_2, p_3) = (q_1(t), q_2(t), q_3(t), 0, 0, 0).$$

with $q_j(t)$ $\quad (j = 1, 2, 3)$ determined by following equations.

$$\frac{\partial q_k}{\partial t_j} = -Y_{jk}(t, q) \quad (k, j = 1, 2, 3).$$

Proposition 6.3. *For the system* (6.3).

(1) *The dimension of solution space over* **C** *is* 4.

(2) *The base of the solution of* (6.3) *is given by*

$$u_i(t) = \int_{r_i} \exp[-\frac{2}{5}\lambda^5 - t_3\lambda^3 - t_2\lambda^2 - (t_1 + \frac{1}{2}t_3^2)\lambda - \frac{1}{2}t_2 t_3]d\lambda \quad (i = 1, 2, 3, 4).$$

where r_i are paths in the complex plane described in Figure 1.

References

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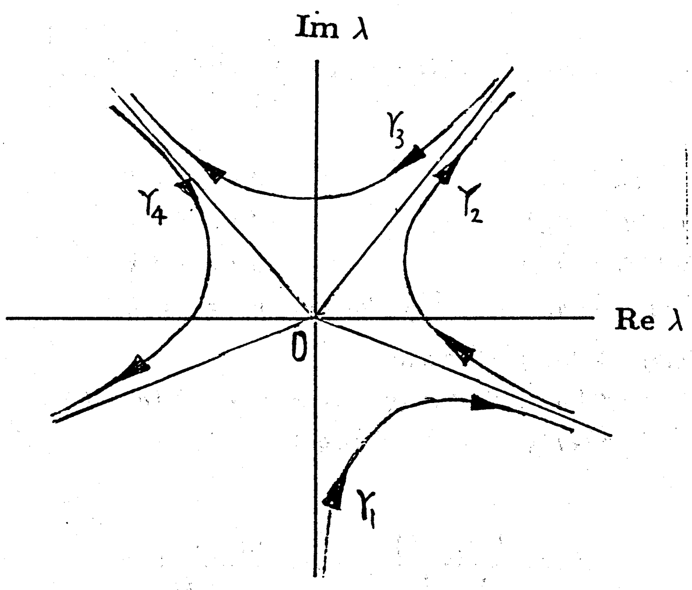

Figure 1: